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Approximating combinatorial contracts with a cardinality constraint

Qinqin Gong[1] · **Ling Gai**[2] · **Yanjun Jiang**[3] · **Yang Lv**[1] · **Ruiqi Yang**[1]

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Abstract

We explore the problem of combinatorial contract design, a subject introduced and studied by Dütting et al. (2023). Previous research has focused on the challenge of selecting an unconstrained subset of agents, particularly when the principal's utility function exhibits XOS or submodular characteristics related to the subset of agents that exert effort. Our study extends this existing line of research by examining scenarios in which the principal aims to select a subset of agents with a specific k-cardinality constraint. In these scenarios, the actions that each agent can take are binary values: effort or no effort. We focus on linear contracts, where the expected reward function is XOS or submodular. Our contribution is an approximation of 0.0197 for the problem of designing multi-agent hidden-action principal-agent contracts with the k-cardinality constraint. This result stands in contrast to the unconstrained setting, where Dütting et al. (2023) achieved an approximation of nearly 0.0039.

Keywords Contracts design · Principal-agent model · Approximation algorithms

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✉ Ruiqi Yang
yangruiqi@bjut.edu.cn

Qinqin Gong
gongqinqin@emails.bjut.edu.cn

Ling Gai
lgai@usst.edu.cn

Yanjun Jiang
jyjmath@163.com

Yang Lv
lvyang@emails.bjut.edu.cn

[1] Institute of Operations Research and Information Engineering, Beijing University of Technology, Beijing 100124, P.R. China

[2] Business School, University of Shanghai for Science and Technology, School of Intelligent Emergency Managemen, University of Shanghai for Science and Technology, Shanghai 200093, P.R. China

[3] School of Mathematics and Statistics Science, Ludong University, Yantai 264025, P.R. China

1 Introduction

Contract theory is an important part of microeconomics. Indeed, contract theory and its core principal-agent model play a pivotal role in the market for services (effort). Its significance is highlighted by the awarding of the 2016 Nobel Prize in Economics to Scientific background on the (2016) for their contributions to contract theory. Contracts based on principal-agent models have been widely applied in various real-world economic scenarios, such as crowdsourcing platforms (Ho et al. 2016), online labor marketplaces (Kaynar and Siddiq 2023), smart contract development (Cong and He 2019), healthcare (Bastani et al. 2016), and the distribution of social goods (Ashlagi et al. 2023; Li et al. 2022).

In the classic hidden-action principal-agent problem (Grossman and Hart 1983; Holmström 1979), a principal aims to incentivize an agent to undertake a costly action in order to complete a task. The principal cannot directly observe the agent's actions but can only observe the outcome, which determines the reward. In this model, a contract specifies the payment that the principal makes to the agent based on the realized outcome. The principal's goal is to design a contract that maximizes her utility, which is defined as the difference between the expected reward and the expected payment. At the same time, the agent chooses an action that maximizes his utility, which is calculated as the expected payment minus the cost incurred from the action. The optimal contract can be determined efficiently in polynomial time by solving linear programming problems appeared in Grossman and Hart (1983).

The principal-agent problem has attracted increasing attention in both economics and computational research. In recent years, the connection between the problem and combinatorial optimization methods has sparked growing interest. This has led to the study of combinatorial contracts, which extend the classic hidden-action principal-agent model. This study mainly includes two settings: multi-action and multi-agent. In the multi-action setting described in Dütting et al. (2021), a principal delegates a costly task to an agent who can undertake any subset of actions. The outcome space is binary, indicating either success or failure. The distribution of outcomes is triggered by any chosen subset of actions. The reward function is defined on the subset of actions chosen by the agent. In the multi-agent setting (Dütting et al. 2023), a principal interacts with multiple agents, each of whom can choose whether to exert effort. The distribution of outcomes is triggered by any subset of agents that exerts effort. The reward function is defined on the subset of agents that exert effort and is examined under various categories, including "complement-free" classes like XOS and submodular functions.

Previous research on combinatorial contract theory has not addressed the problem of selecting a subset of agents under a k-cardinality constraint. This issue has numerous real-world applications. For example, a principal delegates a confidential project to agents. In order to maintain the confidentiality of the project, the principal tends to select only a small number of agents to participate in the actual operations. In a large software development project, a company decides to collaborate with only two outsourcing firms instead of multiple companies. This approach reduces management difficulties and builds a close working relationship between teams. It ensures quick feedback and problem-solving to deliver a high-quality software product on time. In these settings, limiting the number of agents can improve efficiency, control costs, and facilitate collaboration. This motivates us to focus on the problem of choosing a subset of agents under a k-cardinality constraint, which is an important topic in the study of combinatorial contracts.

This paper studies the hidden-action multi-agent contract problem with a k-cardinality constraint. We demonstrate that the optimal contract design problem can be reformulated as a combinatorial subset selection problem with a k-cardinality constraint. As proven in Dütting et al. (2023), finding the optimal contract for submodular reward functions is NP-hard, even in the absence of the k-cardinality constraint. It's worth noting that the complexity of finding the optimal contract remains NP-hard when the input parameter k is on the order of n. We make a significant improvement by developing an approximation algorithm with a factor of 0.0197. This result is applicable when the expected reward function is XOS or submodular. This result stands in contrast to the 0.0039-approximation for the unconstrained setting developed by Dütting et al. (2023).

1.1 Related work

Classic Contracts. Contract theory plays a vital role in microeconomics (Bolton and Dewatripont 2005; Holmström and Milgrom 1987, 1991) and was originally introduced by Ross (1973). The theory and the principal-agent model at its heart have a long research history and have been extensively studied across various fields. The pioneering research on the principal-agent problem includes significant contributions from Holmström (1979) and Grossman and Hart (1983). They showed that a principal can incentivize an agent to complete a task through a well-designed contract. Carroll (2015) and Dütting et al. (2019) showed that linear contracts are max-min optimal. In addition, Dütting et al. (2019) provided a constant-factor approximation for the principal's utility in the optimal contract.

Combinatorial Contracts. In recent years, combinatorial contracts have become an important area of research in contract theory, integrating techniques from combinatorial optimization and approximation algorithms into contract design. It consists of two main settings: multi-action and multi-agent.

In the multi-action scenario, Dütting et al. (2021) showed that the optimal contract can be computed efficiently when the reward function exhibits "gross substitutes" properties as defined by Lehmann et al. (2006). However, if the reward function is submodular, determining the optimal contract becomes NP-hard. Vuong et al. (2024) showed that the optimal contract can be computed efficiently in polynomial time when the reward function is supermodular and the cost function is modular. Dütting et al. (2024) extended this result to scenarios where the cost function is submodular, demonstrating that the optimal contract can still be computed efficiently in polynomial time. These findings represent a significant departure from the NP-hard challenges associated with submodular reward functions, as discussed in earlier work (Dütting et al. 2021).

The multi-agent setting was initially explored by Babaioff et al. (2006), along with its extended version (Babaioff et al. 2012) and subsequent studies (Babaioff et al. 2009; Babaioff et al. 2010). Emek and Feldman (2012) explored the combinatorial agencies model using OR functions and developed an FPTAS. They examined two fundamental Boolean functions: AND, where the project succeeds only if all agents complete their tasks, and OR, where the project succeeds if at least one agent completes their task. Dütting et al. (2023) introduced a combinatorial contract model in which each agent can choose between two binary actions: "effort" and "no effort". The outcome is determined by a probability distribution that depends on the subset of agents exerting effort. When the reward function is XOS or submodular, they provided an approximation algorithm with an approximation ratio of approximately 0.0039 relative to the optimal contract. Vuong et al. (2024) showed that there is no constant factor approximation algorithm when the reward function is supermodular. However, they proposed an additive PTAS for graph-based supermodular valuations with equal costs. Additionally, Castiglioni et al. (2023) introduced a new multi-agent framework. In this model, each agent produces an individual outcome based solely on their actions. The principal observes the individual outcomes and derives a reward based on the collective results. They investigated the computational complexity of finding optimal contracts in relation to the "IR-supermodularity" and "DR-submodularity" of reward functions.

1.2 Organization

The paper is organized as follows: Section 2 defines the problem and model. Section 3 explores the properties of the optimal contract, providing insights for algorithmic design and introducing several key auxiliary lemmas. Sections 4 and 5 together present the constant factor approximation algorithm designed for XOS reward functions, where Section 4 assumes that the threshold of the optimal solution is known, and Section 5 discusses the case where the threshold is unknown. Finally, Section 6 provides a comprehensive summary of our work.

2 Preliminaries

The k-cardinality principal-agent setting with hidden actions. In this model, a principal (referred to as "she") assigns a task to a set of n agents, denoted as $A = \{1, ..., n\}$. Each agent can choose one of two binary actions, indicating whether to exert effort or not. An agent $i \in A$ that exerts effort incurs a cost denoted by $c(i) \in \mathbb{R}_{\geq 0}$.

There is an outcome space denoted as Ω. A subset of agents $S \subseteq A$ exerting effort is associated with a probability distribution $q_S : \Omega \to [0, 1]$ over the outcomes. If agents S exert effort, a stochastic outcome $\omega \in \Omega$ is drawn from q_S, and the principal receives a reward $r(\omega) \in \mathbb{R}_{\geq 0}$. The expected reward of a subset of agents S exerting effort is defined as $f(S) : 2^A \to \mathbb{R}_{\geq 0}$. Formally, $f(S) = \sum_{\omega \in \Omega} q_S(\omega) r(\omega)$, with f assumed to be non-decreasing and monotone. For any $S \subseteq T \in 2^A$, $f(S) \leq f(T)$. Assuming $f(\emptyset) = 0$, the marginal gain of adding agent i to the subset $S \subseteq A$ is $\Delta_f(i|S) = f(S \cup i) - f(S)$.

In the classic model, agents hide their actions, the principal can only observe the resulting stochastic outcome. If the reward of the agents' efforts is fully returned to the principal, the agents may lack the incentive to exert costly effort, a phenomenon known as “moral hazard”. To address this, the principal incentivizes the agents by designing a contract $\mathbf{t} : \Omega \to \mathbb{R}^n_{\geq 0}$ that specifies payments to the agents based on each outcome. Specifically, $\mathbf{t}(\omega) = (t_1(\omega), ..., t_n(\omega))$ denotes the payments to the agents for a realized outcome ω, where $t_i(\omega)$ is the payment to agent i.

A linear contract is represented by a vector $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) \in [0, 1]^n$, which specifies the proportion of the reward allocated to the agents. The payment is expressed as $\mathbf{t}(\omega) = \boldsymbol{\alpha} r(\omega)$, where $t_i(\omega) = \alpha_i r(\omega)$ denotes the payment to agent i for outcome ω. Consequently, the expected payment of agent i is $\alpha_i f(S)$ under the subset of agents S that exert effort.

In this paper, we consider the subset of agents that exert effort to have a size of at most k. The motivation is that the principal can effectively control costs by limiting the number of agents, enhancing efficiency, and reducing risks of underperformance.

Utilities and local optimum. In the contract design problem, the principal aims to design an optimal contract $\boldsymbol{\alpha}$ to maximize her expected utility. Given a linear contract $\boldsymbol{\alpha}$ and a subset of agents S, the principal's expected utility is:

$$u_p(\boldsymbol{\alpha}, S) = \left(1 - \sum_{i \in A} \alpha_i\right) f(S).$$

Meanwhile, each agent seeks to optimize his individual utility by considering whether to exert effort, which needs to be considered in conjunction with the expected payment for exerting effort and the associated cost. The expected utility of agent i is:

$$u_i(\boldsymbol{\alpha}, S) = \alpha_i f(S) - \mathbf{1}_{i \in S} \cdot c(i),$$

where $\mathbf{1}_{i \in S}$ is an indicator function that equals 1 if agent $i \in S$ and 0 otherwise, while $c(i)$ denotes the cost associated with agent i exerting effort.

Given that the reward is distributed among the agents exerting effort, they engage in an induced game and are expected to converge to a local optimum. This local optimum can be characterized using the concept of incentives, employing a similar approach to the widely used “add, delete, and swap” operations found in local search methods for combinatorial optimization. A subset of agents S with a size bounded by k is considered to be *incentivized* by a linear contract $\boldsymbol{\alpha}$ if the following conditions hold:

1. *delete*: $\alpha_i f(S) - c(i) \geq \alpha_i f(S \setminus \{i\})$ for any $i \in S$.
2. *add*: $\alpha_i f(S) \geq \alpha_i f(S \cup \{i\}) - c(i)$ for any $i \notin S$ and $|S| < k$.
3. *swap*: $\alpha_i f(S) \geq \alpha_i f((S \setminus \{j\}) \cup \{i\}) - c(i)$ for any $i \notin S$, $j \in S$ and $|S| = k$.

These conditions mentioned above ensure the benefits of each agent under different strategies. The “delete” condition ensures that any agent i already in the subset S has no incentive to leave. Specifically, the utility of agent i in the subset S, given by $\alpha_i f(S) - c(i)$, must be no less than the utility obtained by exiting the subset, represented as $\alpha_i f(S \setminus \{i\})$. This guarantees that agent i prefer to stay in S rather than leaving. The “add” condition applies to agent i who is not in the subset S while the size of S is less than k. This condition ensures that agent i has no incentive to join the subset. The requirement is that the utility of remaining outside S must be at least as high as the utility that agent i would receive if he joined the subset. Under cardinality constraints ($|S| = k$), the "Swap" condition avoids suboptimal replacements that add/delete alone cannot resolve. While delete and add conditions ensure local stability, the swap operation becomes indispensable under cardinality constraints ($|S| = k$). Without swap, the system may remain trapped in suboptimal equilibria where high-cost agents cannot be displaced by more efficient alternatives even when such replacements would globally improve utility. By systematically evaluating replacements, swap guarantees that the solution space is fully explored, ensuring the principal avoids inefficient lock-in and achieves approximate optimality within the constrained setting.

The optimal contract can be efficiently computed for any given agent subset $S \in 2^A$. Specifically, let

$$\alpha_i = \frac{c(i)}{f(S) - f(S \setminus \{i\})} = \frac{c(i)}{\Delta_f(i|S \setminus \{i\})}$$

for the first case $i \in S$, and let $\alpha_i = 0$ for the other two cases. Each agent's decision problem is whether to exert effort. We note that in the swap operation, it is formally necessary to compare the actions of two agents. However, in practice, the principal only needs to select a subset of agents. Once the subset is chosen, the incentive contract $\boldsymbol{\alpha}$ is determined. The principal must decide which agents to incentivize and optimize the following combinatorial problem:

$$\max_{S \in 2^A : |S| \le k} g(S) = \left(1 - \sum_{i \in S} \frac{c(i)}{\Delta_f(i|S \setminus \{i\})}\right) f(S). \tag{1}$$

Our focus is on optimizing the subset selection problem from the principal's perspective.

In general, g behaves differently from f, even when f is submodular, XOS, or subadditive, as discussed in Dütting et al. (2023).

Reward Function Classes. In this paper, we consider the reward function $f : 2^A \to \mathbb{R}_{\ge 0}$ from the complement-free family (Lehmann et al. 2006):

- *Modular*: The reward function f is modular if $f(S) = \sum_{i \in S} w_i$ for some $w_1, \ldots, w_n \in \mathbb{R}_{\ge 0}$ and any subset $S \in 2^A$, and is also called additive.
- *Submodular*: The reward function f is submodular if for $S \subseteq T$ and $i \in A \setminus T$, $\Delta_f(i|S) \ge \Delta_f(i|T)$.
- *XOS*: The reward function f is XOS if it can be expressed as the maximum of modular functions, i.e., $f(S) = \max_{i \in \{1,\ldots,\ell\}} \sum_{j \in S} v_i(j)$, where $v_i : 2^A \to \mathbb{R}_{\ge 0}$.
- *Subadditive*: The reward function f is subadditive if $f(S \cup T) \le f(S) + f(T)$ for any $S, T \in 2^A$.

The following structural hierarchy can be found in Lehmann et al. (2006), and we restate it here for completeness.

$$modular \subset submodular \subset XOS \subset subadditive.$$

Value and Demand Queries. We assume two primitives for efficiently computing f throughout this paper:

- A *value oracle* for f return $f(S)$ in $O(1)$ for any given S.
- A *demand oracle* for f receives a price vector $p \in \mathbb{R}^n_{\geq 0}$ as input and returns the optimal set $S \in \arg\max_{S \in 2^A : |S| \leq k} f(S) - \sum_{j \in S} p_j$ in $O(1)$ time.

3 Optimizing the principal's utility

The principal's utility is a function of the incentivized set of agents, $g : 2^A \to \mathbb{R}_{\geq 0}$. Let $S^* \in \arg\max_{S \in 2^A : |S| \leq k} g(S)$ be an optimal solution of g under the k-cardinality constraint. By leveraging access to the value and demand oracles, we now present a comprehensive argument for the XOS/submodular reward function.

We partition the agents into two groups using $A(\tau) = \{i \in A | \frac{c(i)}{\Delta_f(i|\emptyset)} \leq \frac{1}{\tau}\}$, where the parameter $\tau > 1$ will be specified in the following sections. Additionally, we denote $\max_{i \in A} g(\{i\})$ as the maximum that incentivizes a single agent. We first establish an upper bound for the maximum $g(S^*)$ by combining the two terms $f(S^* \cap A(\tau))$ and $\max_{i \in A} g(\{i\})$, as stated in the following lemma.

Lemma 1 *If $f : 2^A \to \mathbb{R}_{\geq 0}$ is XOS, then*

$$f(S^* \cap A(\tau)) + \tau \max\left\{0, \max_{i \in A} g(\{i\})\right\} \geq g(S^*).$$

Proof The claim holds directly if $g(S^*) = 0$. We assume without loss of generality that $g(S^*) > 0$. In this case, we obtain

$$\begin{aligned} 0 < g(S^*) &\leq \left(1 - \sum_{i \in S^* \setminus A(\tau)} \frac{c(i)}{\Delta_f(i|S \setminus \{i\})}\right) f(S^*) \\ &\leq \left(1 - \sum_{i \in S^* \setminus A(\tau)} \frac{c(i)}{\Delta_f(i|\emptyset)}\right) f(S^*) \\ &\leq \left(1 - \frac{|S^* \setminus A(\tau)|}{\tau}\right) f(S^*). \end{aligned}$$

The second inequality is derived from the non-negativity of $\frac{c(i)}{\Delta_f(i|S \setminus i)}$ for any $i \in S^*$. The third inequality arises from the subadditivity of f. The last inequality follows by the definition of $A(\tau)$.

Therefore, we derive $|S^* \setminus A(\tau)| < \tau$. Assume $|S^* \setminus A(\tau)| = m$ satisfies $m < \tau$. If $m = 0$, then $S^* \subseteq A(\tau)$, which implies that $f(S^* \cap A(\tau)) \geq g(S^* \cap A(\tau)) = g(S^*)$. Consider the general case where we assume that $S^* \setminus A(\tau) = \{i_1, ..., i_m\}$. Then the following inequalities are hold:

$$g(S^*) \leq \left(1 - \sum_{i \in S^* \setminus A(\tau)} \frac{c(i)}{\Delta_f(i|S \setminus \{i\})}\right) f(S^*)$$

$$
\begin{aligned}
&\leq \left(1 - \sum_{i \in S^* \setminus A(\tau)} \frac{c(i)}{f(\{i\})}\right) (f(S^* \cap A(\tau)) + f(S^* \setminus A(\tau)))\\
&\leq g(\{i_1\}) + \cdots + g(\{i_m\}) + f(S^* \cap A(\tau))\\
&\leq \tau \max_{i \in A} g(\{i\}) + f(S^* \cap A(\tau)),
\end{aligned}
$$

where the second inequality holds due to the subadditivity of f, and the third inequality follows from the non-negativity of $\frac{c(i)}{f(i)}$ for any $i \in S^*$. To obtain the last inequality, it concludes that $g(\{i_j\}) \leq \max_{i \in A} g(\{i\})$ for each $i_j \in S^* \cap A(\tau)$ where $j \in \{1, ..., m\}$. □

The two lemmas below establish a connection between the principal's utility and the expected reward.

Lemma 2 *(Dütting et al. 2023) If $f : 2^A \to \mathbb{R}_{\geq 0}$ is XOS, then for each $S \subseteq S^*$, we have*

$$\sum_{i \in S} \sqrt{c(i)} \leq \sqrt{f(S)}.$$

Note that this result holds regardless of cardinality constraints, as the XOS property ensures the inequality $\sum_{i \in S} \sqrt{c(i)} \leq \sqrt{f(S)}$ for any subset S^*.

Lemma 3 *If $f : 2^A \to \mathbb{R}_{\geq 0}$ is XOS, then for any agent subset $S \subseteq A$ with a reward $f(S) > 0$ and parameter $\rho > 1$, the following relationship holds:*

$$g(S) \geq \left(1 - \frac{1}{\rho}\right) f(S)$$

if $\Delta_f(i|S \setminus \{i\}) \geq \sqrt{\rho c(i) f(S)}$ for any $i \in S$.

Proof Given that $\Delta_f(i|S \setminus \{i\}) \geq \sqrt{\rho c(i) f(S)}$ for any $i \in S$, we can derive the following inequalities:

$$\sum_{i \in S} \frac{c(i)}{\Delta_f(i|S \setminus \{i\})} \leq \sum_{i \in S} \frac{\Delta_f(i|S \setminus \{i\})}{\rho f(S)} = \frac{\sum_{i \in S} \Delta_f(i|S \setminus \{i\})}{\rho f(S)} \leq \frac{1}{\rho},$$

where the second inequality follows from Lemma 2.1 in Dütting et al. (2023). Thus we have

$$g(S) = \left(1 - \sum_{i \in S} \frac{c(i)}{\Delta_f(i|S \setminus \{i\})}\right) f(S) \geq \left(1 - \frac{1}{\rho}\right) f(S).$$

□

Lemma 3 generalizes the result from Lemma 3.4 in Dütting et al. (2023) by introducing a parameter $\rho > 1$. The lemma reduces to the case presented in Dütting et al. (2023) when $\rho = 2$. We now explore a novel scaling property of XOS functions.

Lemma 4 *(Dütting et al. 2023) Let* $f : 2^A \to \mathbb{R}_{\geq 0}$ *be an XOS function. Given any* $T \in 2^A$*, and parameters* $\delta \in (0, 1]$ *and* $\Psi \in [0, f(T))$*, there exists a polynomial time algorithm called Scaling-Set-XOS-Algorithm* (f, T, Ψ, δ) *that utilizes the value oracle to access* f *and returns a set* U *satisfying*

$$(1-\delta)\Psi \leq f(U) \leq \Psi + \max_{i \in T} f(\{i\}),$$

and

$$\Delta_f(i|U \setminus \{i\}) \geq \delta \Delta_f(i|T \setminus \{i\})$$

for any $i \in U$.

Note that the Scaling-Set-XOS-Algorithm (f, T, Ψ, δ) is originally introduced by Dütting et al. (2023), and we restated it here for completeness. The details are summarized in Algorithm 1.

Algorithm 1 Scaling-Set-XOS-Algorithm(f, T, Ψ, δ) (Dütting et al. 2023)

```
Input: An XOS function $f : 2^A \to \mathbb{R}_{\geq 0}$, parameters $\Psi \in (0, f(T))$ and $\delta \in (0, 1]$
Output: A set $U \subseteq T$
Define $T_0$ as the smallest subset of $T$ for which $f(T_0) = f(T)$
for $t = 1, ..., |T_0|$ do
    Set $i_t \in \arg\min_{i \in T_{t-1}} \frac{\Delta_f(i|T_{t-1} \setminus \{i\})}{\Delta_f(i|T_0 \setminus \{i\})}$
    Set $T_t \leftarrow T_{t-1} \setminus \{i_t\}$
    Set $\delta_t = \frac{\Delta_f(i_t|T_{t-1} \setminus \{i\})}{\Delta_f(i_t|T_0 \setminus \{i\})}$
end for
Set $j^* = \min\{j | f(T_j) \leq \Psi\}$
Set $k^* = \min\left\{k | \frac{f(T_k)}{f(T_{j^*})} \leq 1 - \delta\right\}$
Set $t^* = \arg\max_{t \in \{j^*, ..., k^*\}} \delta_t$
return $U \leftarrow T_{t^*-1}$
```

4 Knowing $f(S^* \cap A(\tau))$ helps

This section presents an algorithm that estimates $f(S^* \cap A(\tau))$, outlined in Algorithm 2.

By appropriately configuring a price to access the demand oracle and leveraging the results of Lemma 4, we determine a contract that provides a constant-factor approximation to $f(S^* \cap A(\tau))$. The main results are summarized in the theorem below.

Theorem 1 *If $f(S^* \cap A(\tau)) \geq \nu \geq \frac{1}{1+\varepsilon} f(S^* \cap A(\tau))$, then by utilizing the value and demand oracles, Algorithm 2 returns a set U in polynomial time such that*

$$g(U) \geq \max\left\{\left(1 - \frac{\beta(\beta-1)}{\delta^2}\right)(1-\delta)\left[\left(1-\frac{1}{\beta}\right)\nu - \max_{i\in A(\tau)} f(\{i\})\right], 0\right\}.$$

Proof We will consider the case where $\Psi > 0$; otherwise, the claim holds by choosing $U = \emptyset$. In this case, we have the following inequalities:

$$\begin{aligned} f(T^*) &\geq f(T^*) - \sum_{i\in T^*} p(i) \geq f(S^* \cap A(\tau)) - \sum_{i \in S^* \cap A(\tau)} p(i) \\ &= f(S^* \cap A(\tau)) - \frac{1}{\beta} \sum_{i\in S^*\cap A(\tau)} \sqrt{\nu c(i)} \\ &\geq f(S^* \cap A(\tau)) - \frac{1}{\beta}\sqrt{f(S^* \cap A(\tau))}\left(\sum_{i\in S^*\cap A(\tau)} \sqrt{c(i)}\right) \\ &\geq \left(1-\frac{1}{\beta}\right) f(S^* \cap A(\tau)) \\ &\geq \left(1-\frac{1}{\beta}\right)\nu - \max_{i\in A(\tau)} f(\{i\}) = \Psi, \end{aligned}$$

where the first inequality follows by the non-negativity of $p(i)$, the second inequality holds due to $T^* \in \arg\max_{T\in 2^{A(\tau)}:|T|\leq k}\left\{f(T) - \sum_{i\in A(\tau)} p(i)\right\}$,

Algorithm 2 Estimate-Based Approximation Algorithm

```
Input: An XOS function $f : 2^A \to \mathbb{R}_{\geq 0}$, cost function $c : A \to \mathbb{R}_{\geq 0}$, parameters $\varepsilon \in (0,1)$, $\beta > 1$, and
    $1 \geq \delta > \sqrt{\beta(\beta-1)}$; an estimate $\nu$ of $f(S^* \cap A(\tau))$ satisfying
        $f(S^* \cap A(\tau)) \geq \nu \geq \frac{1}{1+\varepsilon} f(S^* \cap A(\tau))$
Output: A set $U \subseteq A(\tau)$
Set $\Psi \leftarrow \left(1-\frac{1}{\beta}\right)\nu - \max_{i\in A(\tau)} f(\{i\})$
Set $p(i) \leftarrow \frac{1}{\beta}\sqrt{\nu c(i)}$ for each $i \in A(\tau)$
Let $T^* \in \arg\max_{T\in 2^{A(\tau)}:|T|\leq k} \left\{f(T) - \sum_{i\in A(\tau)} p(i)\right\}$
if $\Psi \in (0, f(T^*))$ then
    $U \leftarrow$ Scaling-Set-XOS-Algorithm $(f, T^*, \Psi, \delta)$
else
    $U \leftarrow \emptyset$
end if
```

the third inequality follows from $\nu \leq f(S^* \cap A(\tau))$, the fourth inequality is derived from Lemma 2, and the last inequality holds by the non-negativity of $\max_{i\in A(\tau)} f(\{i\})$.

According to Lemma 4, there exists a scaling set U satisfying

$$f(U) \geq (1-\delta)\Psi = (1-\delta)\left[\left(1-\frac{1}{\beta}\right)\nu - \max_{i \in A(\tau)} f(\{i\})\right]$$

and

$$f(U) \leq \Psi + \max_{i \in T^*} f(\{i\}) \leq \left(1-\frac{1}{\beta}\right)\nu. \tag{2}$$

For any $i \in U$, we have

$$\begin{aligned} \Delta_f(i|U \setminus \{i\}) &\geq \delta \Delta_f(i|T^* \setminus \{i\}) \\ &\geq \delta p(i) = \frac{\delta}{\beta}\sqrt{\nu c(i)} \\ &\geq \sqrt{\frac{\delta^2}{\beta(\beta-1)} c(i) f(U)}. \end{aligned}$$

The first inequality holds by Lemma 4. The second inequality follows from $\Delta_f(i|T^* \setminus \{i\}) \geq p(i)$ for all $i \in T^*$, as T^* is a demand set over the set $A(\tau)$. The third inequality is derived from Inequality (2).

Let $\rho = \delta^2/(\beta^2-\beta)$. By Lemma 3, we obtain the following inequality:

$$g(U) \geq \left(1-\frac{1}{\rho}\right) f(U) \geq \left(1-\frac{\beta(\beta-1)}{\delta^2}\right)(1-\delta)\left[\left(1-\frac{1}{\beta}\right)\nu - \max_{i \in A(\tau)} f(\{i\})\right].$$

□

Algorithm 3 Approximating Optimal Contract

```
Input: An XOS function $f : 2^A \to \mathbb{R}_{\geq 0}$, cost function $c : A \to \mathbb{R}_{\geq 0}$, parameters $l \in \mathbb{Z}$ and $\varepsilon \in (0, 1)$
Output: A set $S \in 2^A$
Set $\mathcal{C} = \{\{i\} | i \in A\} \cup \{\emptyset\}$
Set $A(\tau) = \left\{ i \in A | \frac{c(i)}{f(\{i\})} \leq \frac{1}{\tau} \right\}$
Let $(1+\varepsilon)^l = \left(1-\frac{1}{\tau}\right) \max_{i \in A(\tau)} f(\{i\})$
for $j \in \{0, ..., \log_{1+\varepsilon} \frac{k\tau}{\tau-1}\}$ do
    Let $\nu_j = (1+\varepsilon)^{l+j}$
    Set $U_{\nu_j} \leftarrow$ the output of Algorithm 1 with $\nu = \nu_j$
    Set $\mathcal{C} \leftarrow \mathcal{C} \cup U_{\nu_j}$
end for
return $S \leftarrow \arg\max_{S \in \mathcal{C}} g(S)$
```

5 Guess $f(S^*) \cap A(\tau)$ for approximation

In this section, we introduce a geometric estimation for the value of $f(S^* \cap A(\tau))$. By integrating the estimation process with the subroutine described in Section 4, we achieve the desired approximation guarantee for our cardinality-constrained principal-agent contract design problem.

The details are presented in Algorithm 3, which outlines the key steps involved. Our main results are summarized in the following theorem.

Theorem 2 *By utilizing the value and demand queries, with* $\varepsilon = 0.01$*, Algorithm 3 achieves a* 0.0197*-approximation for Problem (1) in polynomial time.*

Proof We divide our argument into two cases with a fixed parameter $\gamma > 1$. **Case 1.** If $f(S^* \cap A(\tau)) \leq \gamma \max\{0, \max_{i\in A} g(\{i\})\}$, then by Lemma 1, we obtain

$$g(S^*) \leq f(S^* \cap A(\tau)) + \tau \max\{0, \max_{i\in A} g(\{i\})\} \leq (\gamma + \tau) \max\{0, \max_{i\in A} g(\{i\})\}.$$

Case 2. Assume $f(S^* \cap A(\tau)) > \gamma \max\{0, \max_{i\in A} g(\{i\})\}$. By applying Lemma 1, we derive

$$\begin{aligned} g(S^*) &\leq f(S^* \cap A(\tau)) + \tau \max\left\{0, \max_{i\in A} g(\{i\})\right\} \\ &\leq f(S^* \cap A(\tau)) + \frac{\tau}{\gamma} f(S^* \cap A(\tau)) \\ &\leq \left(1 + \frac{\tau}{\gamma}\right) f(S^* \cap A(\tau)). \end{aligned} \tag{3}$$

For any $i \in A(\tau)$, it concludes that

$$g(\{i\}) = \left(1 - \frac{c(i)}{f(\{i\})}\right) f(\{i\}) \geq \left(1 - \frac{1}{\tau}\right) f(\{i\}).$$

Therefore we obtain

$$f(S^* \cap A(\tau)) \geq \max_{i\in A} g(\{i\}) \geq \left(1 - \frac{1}{\tau}\right) \max_{i\in A(\tau)} f(\{i\}).$$

By the subadditivity of f, we have

$$f(S^* \cap A(\tau)) \leq \sum_{i\in S^*\cap A(\tau)} f(\{i\}) \leq k \max_{i\in A(\tau)} f(\{i\}).$$

There exist a unique $j^* \in \{0, ..., \log_{1+\varepsilon} \frac{k\tau}{\tau-1}\}$ such that $\nu_{j^*} \leq f(S^* \cap A(\tau)) \leq (1+\varepsilon)\nu_{j^*}$.

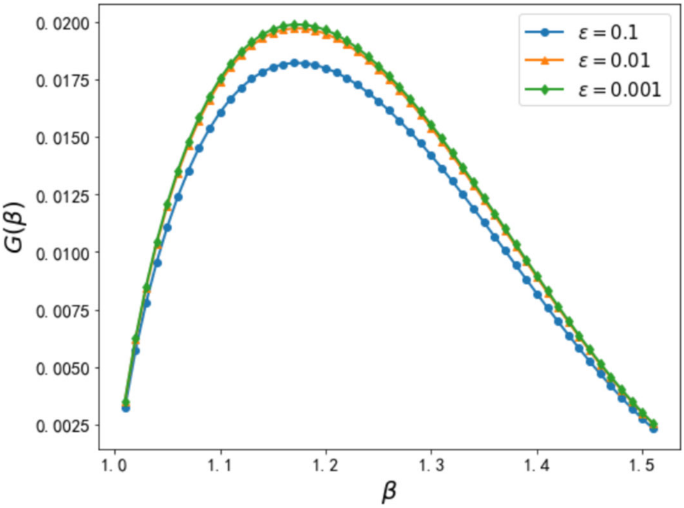

Fig. 1 The curves represent the variation of the approximation ratio $G(\beta)$ with the parameters $\beta \in (1, 1.5)$ and $\varepsilon = 0.001, 0.01$, and 0.1

For simplicity, let $a = \left(1 - \frac{\beta(\beta-1)}{\delta^2}\right)(1-\delta)$, $b = 1 - 1/\beta$. We now analyze the principal's utility value on the set $U_{\nu_{j^*}}$ as follows.

$$\begin{aligned} g(U_{\nu_{j^*}}) &\geq \left(1 - \frac{\beta(\beta-1)}{\delta^2}\right)(1-\delta)\left[\left(1 - \frac{1}{\beta}\right)\nu_{j^*} - \max_{i \in A(\tau)} f(\{i\})\right] \\ &\geq a\left[\frac{b}{1+\varepsilon} f(S^* \cap A(\tau)) - \max_{i \in A(\tau)} f(\{i\})\right] \\ &\geq a\left[\frac{b}{1+\varepsilon} f(S^* \cap A(\tau)) - \frac{\max_{i \in A} g(\{i\})}{1 - 1/\tau}\right] \\ &\geq a\left[\frac{b}{1+\varepsilon} f(S^* \cap A(\tau)) - \frac{f(S^* \cap A(\tau))}{\gamma(1 - 1/\tau)}\right] \\ &\geq \frac{a\gamma}{\gamma+\tau}\left(\frac{b}{1+\varepsilon} - \frac{1}{\gamma(1-1/\tau)}\right) g(S^*), \end{aligned}$$

where the first inequality follows from the claim in Theorem 1, the second inequality holds because $\nu_{j^*} \geq f(S \cap A(\tau))/(1+\varepsilon)$, the third inequality is derived from $g(\{i\}) \geq \left(1 - \frac{1}{\tau}\right) f(\{i\})$ for every $i \in A(\tau)$, the fourth inequality results from $f(S \cap A(\tau)) > \gamma \max\{0, \max_{i \in A} g(\{i\})\}$, and the last inequality holds due to Inequality (3).

Let the approximation ratios in the above two cases be equal, i.e.,

$$\frac{1}{\gamma+\tau} = \frac{a\gamma}{\gamma+\tau}\left(\frac{b}{1+\varepsilon} - \frac{1}{\gamma(1-1/\tau)}\right)$$

Table 1 Comparison of performance guarantees for combinatorial principal-agent contract problems. The approximation ratios (Dütting et al., 2023) are influenced only by the parameter ε, whereas our results are improved by additional parameters, namely β and δ. δ^* denotes the optimal δ that maximizes $G(\beta)$ for any given β and ε

ε	β	δ^*	Approx. Ratios	
			Ours	Dütting et al. (2023)
0.1	1.1700	0.6458	0.0182	0.0035
	1.1732	0.6498	0.0182	
	1.1801	0.6582	0.0182	
	1.1894	0.6692	0.0181	
	1.1839	0.6628	0.0181	
	1.1923	0.6726	0.0182	
	1.1998	0.6812	0.0180	
0.01	1.1700	0.6458	0.0197	0.0038
	1.1732	0.6498	0.0197	
	1.1801	0.6582	0.0197	
	1.1894	0.6692	0.0197	
	1.1839	0.6628	0.0196	
	1.1923	0.6726	0.0196	
	1.1998	0.6812	0.0195	
0.001	1.1700	0.6458	0.0199	0.0039
	1.1732	0.6498	0.0199	
	1.1801	0.6582	0.0199	
	1.1894	0.6692	0.0198	
	1.1839	0.6628	0.0198	
	1.1923	0.6726	0.0198	
	1.1998	0.6812	0.0196	

and we obtain

$$\begin{aligned}\gamma &= \left(1 + \frac{a\tau}{\tau - 1}\right)\left(\frac{1+\varepsilon}{ab}\right)\\ &= \frac{1+\varepsilon}{(1 - \frac{\beta(\beta-1)}{\delta^2})(1-\delta)(1-\frac{1}{\beta})} + \frac{(1+\varepsilon)\tau}{(\tau - 1)(1 - \frac{1}{\beta})}.\end{aligned}$$

Denote $G_\delta(\tau) = \frac{1}{\gamma + \tau}$ as the parameterized approximation ratio with respect to τ. Its derivative function with respect to τ is given by

$$G'_\delta(\tau) = \frac{-\tau^2 + 2\tau + \frac{1+\varepsilon}{1-1/\beta} - 1}{\left[\tau^2 + \left(\frac{1+\varepsilon}{(1-\beta(\beta-1)/\delta^2)(1-\delta)(1-1/\beta)} + \frac{1+\varepsilon}{1-1/\beta} - 1\right)\tau - \frac{1+\varepsilon}{(1-\frac{\beta(\beta-1)}{\delta^2})(1-\delta)(1-1/\beta)}\right]^2},$$

where $G'_\delta(\tau) \geq 0$ when $\tau \in \left[1, 1 + \sqrt{\frac{1+\varepsilon}{1-1/\beta}}\right]$, and $G'_\delta(\tau) < 0$ when $\tau \in \left(1 + \sqrt{\frac{1+\varepsilon}{1-1/\beta}}, +\infty\right)$. According to the conditions for extremum, the maximum of

$G_\delta(\tau)$ can be attained at

$$\tau = 1 + \sqrt{\frac{1+\varepsilon}{1-1/\beta}}.$$

We further need to optimize the following parameterized approximation ratio:

$$G(\beta) = \max_{\delta \in (\sqrt{\beta(\beta-1)},1]} \frac{1}{\frac{1+\varepsilon}{(1-\beta(\beta-1)/\delta^2)(1-\delta)(1-1/\beta)} + \left(\sqrt{\frac{1+\varepsilon}{1-1/\beta}}+1\right)^2}.$$

The value of $G(\beta)$ varies with the parameters β and ε, as illustrated in Figure 1.

We summarized the results computed using the parameters β and δ in Table 1, comparing them with the results from Dütting et al. (2023).

We directly achieve an approximation ratio of approximately 0.0197 by setting $\beta = 1.1734$ and $\delta = 0.64578$ when $\varepsilon = 0.01$. □

6 Conclusion

This paper studies the principal multi-agent contract design problem with a k-cardinality constraint. The principal aims to design a contract that maximizes her expected utility by incentivizing a subset of agents to exert effort. Our approach transforms the optimal contract design problem into an agent subset selection problem with a fixed cardinality constraint of size k. By examining the combinatorial structures inherent in the subset selection problem and designing an approximation algorithm, we obtain a 0.0197-approximation solution. This is particularly applicable when the reward function is XOS or submodular. Our result represents a significant improvement over the nearly 0.0039-approximation previously offered by Dütting et al. (2023), which addressed a similar problem without the k-cardinality constraint. There are still many open questions that deserve to be investigated. We may consider studying the principal multi-agent contract design problem within a Bayesian framework. Additionally, in some real-world cases, the reward function may not be monotonic. Therefore, we can extend our study to cases where the monotonicity is not guaranteed and explore the optimal contract design problem.

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Declarations

Conflicts of Interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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